

Optimal control for a vibrating string with variable axial load and damping gain

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Abstract: We consider an optimal control problem for a vibrating string with multiplicative distributive control. Multiplicative control is the coefficient $u(x, t)$ of velocity term y_t . Using multiplicative control, we bring the state solution to the desired profile under a quadratic cost of control. We prove the existence of optimal control. Criterion for characterization of the unique optimal control is provided in the form of optimality system.

Keywords: Optimal Control; Distributed Systems; Nonlinear Systems and Control, Viscous damping coefficient, Bilinear control.

1. INTRODUCTION

Various engineering applications require prediction of outputs governed by partial differential equations. Many real-life applications of optimal control problems with constraints in the form of partial differential equations (PDEs) are modeled using hyperbolic equations. Hyperbolic equations describe transport processes. Because of their nature hyperbolic equations are able to transport discontinuities of initial or boundary conditions into the domain on which the solution lives or to develop discontinuities in the presence of smooth data, these problems constitute a severe challenge for both theory and numerics of PDE constrained optimization.

In structural dynamics, dynamic instability created due to parametric resonance is an important factor. For example, it is believed that instability caused by parametric resonance is the reason for the famous Tacoma bridge collapse in 1940 (Berdichersky et al. (1999)). We can control this phenomenon by suitable control of the coefficients. Control of coefficients has been studied and applied in structures governed by elliptic problems (Lurie (1999)), well developed literature is available in this case. However, the effect produced by bilinear control for hyperbolic problems is not studied rigorously (Lurie (1999)-Lurie (2003)). Optimal control of systems governed by hyperbolic equations is of special importance for the active control of structural systems. The field of structural control has been an active research area for a number of years. However, most of the studies in this area considered specific structures such as wings (Wang (1988)), beams (Pan and Hansen (1993)) and plates (Bruch (Jr.)). Even though these studies provide solutions for many particular cases, the theoretical foundations of the subject aimed specifically at problems arising in structural mechanics have not received much attention. Theoretical studies such as the ones in Basile and

Mininni (1990), Fattorini (1993) considered the optimal control problems in abstract settings leaving a gap between the theory and applications. In particular, optimal control studies relating the theory directly to the solution method have been scarce. However, one such application is given in Bruch (Jr.) (1995) for a structural vibration problem governed by a single hyperbolic equation. Another application is given herein for a vibration problem governed by a system of hyperbolic equations. The maximum principle (Bruch (Jr.) (2006)) was used to construct explicit solutions for an optimal control problem involving a distributed parameter structure governed by a system of hyperbolic differential equations. Numerical methods for solving such problems have not received much attention. Sonawane et al. (2013) considered the numerical treatment of the optimal control problems for bilinear hyperbolic systems.

In this paper, we study the optimal control problem for a dynamic system governed by a hyperbolic equation with multiplicative control $u(x, t)$ which is a coefficient of velocity term y_t . The control u can be interpreted as the gain of the viscous (motion-activated) damping acting upon the string at point x and time t . Bradley, Lenhart and other treated the optimal control problem with multiplicative control for the plate equation without velocity term in Lenhart and Bradley (1994) and with velocity term in Bradley et al. (1999). Liang (1999), Lenhart et al. (1999) treated optimal control problem with multiplicative control without velocity term for wave equation, where as Lenhart and Liang (2000) treated similar problem with velocity term for wave equation. Belinskiy and Lenhart (2001) treated the optimal control problem with multiplicative controls for a wave equation with viscous damping. In Lenhart and Liang (2000), a control function is a

function of time variable t only, where as in Belinskiy and Lenhart (2001) it is a function of a space variable x only.

Let $\Omega = (0, l)$ and $T > 0$. Consider the following initial-boundary value problem for a hyperbolic equation:

$$\begin{aligned} w_{tt} &= w_{xx} - uw_t + vw + f, \text{ in } Q = \Omega \times (0, T), \\ w &= 0 & x = 0, t \in (0, T), \\ w &= 0 & x = l, t \in (0, T), \\ w &= w_0(x), & t = 0, x \in \Omega, \\ w_t &= w_1(x), & t = 0, x \in \Omega, \end{aligned} \quad (1)$$

where $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$, $f \in L^2(Q)$, $v \in L^\infty(Q)$. $u \in H_0^1(Q)$ is a multiplicative (also known as bilinear) distributive control. We assume that the control functions belong to the set of admissible controls:

$$U = \{u \in H_0^1(Q) | 0 \leq |u| \leq M\}, \quad (2)$$

where M is a constant.

Consider the objective function:

$$J(u) = \frac{\alpha}{2} \int_Q (w - z)^2 dx dt + \frac{\beta}{2} \int_Q [(u_x)^2 + (u_t)^2] dx dt \quad (3)$$

where $z \in L^2(Q)$ is given target function. This objective function combines the energy of control and the measure of the “closedness” to the desired profile z . We want to minimize the function over the set of admissible controls U , i.e. to characterize u^* such that

$$J(u^*) = \min_{u \in U} J(u).$$

We shall state and prove the existence of optimal control functions. We, then, shall give a characterization for control $u(x, t)$ which is a coefficient of a velocity term for given $v(x, t)$. The rest of the paper is organized as follows. In section 2, we prove the well-posedness of our problem and obtain priori estimates. Then, we show that there exist optimal control function which minimizes the cost functional. In section 3, we derive the optimality system by differentiating the cost functional with respect to control function u and give the characterization of an optimal control through an elliptic variational inequality.

2. EXISTENCE OF OPTIMAL CONTROLS

The weak solution of equation (1) is defined as follows:

Definition 2.1. Given $u \in U$, we say that a function $w \in C([0, T]; H_0^1(\Omega))$, with $w_t \in C([0, T]; L^2(\Omega))$, $w_{tt} \in C([0, T]; H^{-1}(\Omega))$ is a weak solution of the problem (1) if

- for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$,

$$\langle w_{tt}, \phi \rangle + B[w, \phi] + \int_\Omega uw_t \phi dx = \int_\Omega f \phi dx,$$

- $w(x, 0) = w_0(x)$, $w_t(x, 0) = w_1(x)$;
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1}(\Omega)$, $H_0^1(\Omega)$, and $B[w, \phi; t] = \int_\Omega w_x \phi_x dx - \int_\Omega vw \phi dx$ is a bilinear form.

Set the notational conventions as follows:

$$\begin{aligned} \mathcal{H} &= H_0^1(\Omega) \times L^2(\Omega), \\ \tilde{w} &= (w, w_t). \end{aligned}$$

First, we state and prove the existence, uniqueness, regularity and priori estimates for weak solutions of equation (1). We, here assume that the control functions are given to study the dynamics of the solutions.

Theorem 2.1. Assume that $w_0 \in H^2(\Omega)$, $w_1 \in H_0^1(\Omega)$, $u \in U \cap H^1(Q)$, $v \in C^1(Q)$, and $f, f_t \in L^2(Q)$. Then the problem (1) has a unique weak solution w satisfying $w \in C([0, T]; H^2(\Omega))$, $w_t \in C([0, T]; H^1(\Omega))$ and $w_{tt} \in C([0, T]; L^2(\Omega))$. Moreover w satisfies the bound estimate

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left(\|w(t)\|_{H_0^1(\Omega)} + \|w_t(t)\|_{L^2(\Omega)} \right) \\ &\quad + \|w_{tt}(t)\|_{L^2([0, T]; H^{-1}(\Omega))} \\ &\leq C \left(\|f\|_{L^2([0, T]; L^2(\Omega))} + \|w_0\|_{H_0^1(\Omega)} + \|w_1\|_{L^2(\Omega)} \right), \quad (4) \end{aligned}$$

where C is a constant which depends only on Ω and T .

Proof. We will prove the theorem in three steps. In first step we will prove the existence of a unique weak solution. Regularity of the weak solution will be proved in second step 2. And, estimate (4) will be obtained in third step.

Step 1. We write the equation (1) in the following semigroup form

$$\begin{pmatrix} w \\ w_t \end{pmatrix}_t = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix} \begin{pmatrix} w \\ w_t \end{pmatrix} + \begin{pmatrix} 0 \\ -uw_t + vw + f \end{pmatrix}, \quad (5)$$

$$\tilde{w}(0) = \tilde{w}_0 = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

Define the operator $A : [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \rightarrow \mathcal{H}$ by

$$A\tilde{w} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix} \tilde{w}$$

The domain of A , $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ is clearly dense in $\mathcal{H}$. Equation (5) can be written as

$$\frac{d}{dt} \tilde{w}(t) = A\tilde{w}(t) + \hat{B}(\tilde{w})(t), \quad (6)$$

where

$$\hat{B}(\tilde{w}) = \begin{pmatrix} 0 \\ -uw_t + vw + f \end{pmatrix}.$$

Note that A is skew adjoint (Pazy (1983)), i.e., $A^* = -A$. Thus $D(A) = D(A^*)$, and hence it generates a unitary group on $\mathcal{H}$.

Due to the semigroup form (6), a weak solution should be of the form

$$\tilde{w}(t) = e^{At} \tilde{w}_0 + \int_0^t e^{A(t-\tau)} \hat{B}(\tilde{w})(\cdot, \tau) d\tau,$$

where e^{At} represents the semigroup generated by A .

Define a map S as,

$$S\tilde{w}(t) = e^{At} \tilde{w}_0 + \int_0^t e^{A(t-\tau)} \hat{B}(\tilde{w})(\cdot, \tau) d\tau.$$

We will prove that the map S has a unique fixed point

$$S\tilde{w}(t) = \tilde{w}(t)$$

in $C([0, T_0]; \mathcal{H})$, where T_0 is to be chosen.

First, we prove that S is bounded:

$$\begin{aligned}
& \|S\tilde{w}\|_{C([0,T_0];\mathcal{H})} \leq \|e^{At}\tilde{w}_0\|_{C([0,T_0];\mathcal{H})} \\
& \quad + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)}\hat{B}(\tilde{w})(\cdot, \tau)\|_{\mathcal{H}} d\tau \\
& \leq \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)}[-u(\cdot, \tau)w_t(\cdot, \tau) + v(\cdot, \tau)w(\cdot, \tau)]\|_{L^2(\Omega)} d\tau \\
& \quad + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)}f(\cdot, \tau)\|_{L^2(\Omega)} d\tau \\
& \quad + \|e^{At}\tilde{w}_0\|_{C([0,T_0];\mathcal{H})} \\
& \leq \sup_{0 \leq t \leq T_0} \int_0^t \|-u(\cdot, \tau)w_t(\cdot, \tau) + v(\cdot, \tau)w(\cdot, \tau)\|_{L^2(\Omega)} d\tau \\
& \quad + \sup_{0 \leq t \leq T_0} \int_0^t \|f(\cdot, \tau)\|_{L^2(\Omega)} d\tau + \|\tilde{w}_0\|_{\mathcal{H}},
\end{aligned}$$

as $\|e^{At}\| = 1$. Since $\|u\|_{H^1(Q)} \leq M$ we obtain

$$\begin{aligned}
\|S\tilde{w}\|_{C([0,T_0];\mathcal{H})} & \leq \|\tilde{w}_0\|_{\mathcal{H}} + (C_{01} + C_{02})T_0\|\tilde{w}\|_{C([0,T_0];\mathcal{H})} \\
& \quad + T_0\|f\|_{L^\infty(Q)}, \quad (7)
\end{aligned}$$

where constant C_{01} depends only on M . Hence S bounded.

Next, we show that S is contractive: For any $\tilde{p}, \tilde{q} \in C([0, T_0]; \mathcal{H})$, we have

$$\|S\tilde{p} - S\tilde{q}\|_{C([0,T_0];\mathcal{H})} \leq (M_v + M)T_0\|\tilde{p} - \tilde{q}\|_{C([0,T_0];\mathcal{H})}, \quad (8)$$

where M_v is a bound for v .

By selecting $T_0 < \frac{1}{(M+M_v)}$, we conclude that S is contractive for $t \leq T_0$.

From (7)-(8), we conclude that S is bounded and contractive on $C([0, T_0]; \mathcal{H})$. Thus, by the contraction mapping theorem, there exists a unique fixed point for the map S on $C([0, T_0]; \mathcal{H})$.

Now, we extend above result to a solution on $[T_0, 2T_0]$ by selecting new initial value as

$$\tilde{w}_{T_0} = \tilde{w}(T_0) \in \mathcal{H}.$$

By applying contraction mapping theorem second time, we have a unique solution on $C([T_0, 2T_0]; \mathcal{H})$. Repeating the process finite number of times, we conclude the existence of a unique solution to (1) with $\tilde{w} \in C([0, T]; \mathcal{H})$.

Step 2. Let us select a sequence of approximations by selecting smooth functions $\sigma_j = \sigma_j(x)$, ($j = 1, 2, \dots$) such that

$$\{\sigma_j\}_{j=1}^\infty \text{ is a basis of } H_0^1(\Omega)$$

and

$$\{\sigma_j\}_{j=1}^\infty \text{ is an orthonormal basis of } L^2(\Omega),$$

where $\{\sigma_j\}$ are eigenvectors of $-\Delta$ (corresponding to the eigenvalues λ_j). For integer m , write

$$w_m(t) \equiv \sum_{j=1}^m d_j^m(t) \sigma_j(s) \quad (9)$$

where $d_j^m(t)$ satisfies

$$\begin{aligned}
d_j^m(0) &= (w_0(x), \sigma_j)_{L^2}, \quad (d_j^m)_t(0) = (w_1(x), \sigma_j)_{L^2} \\
&\quad (j = 1, 2, \dots, m) \quad (10)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} (w_m)_{tt}(t) \sigma_j dx + \int_{\Omega} u(x, t) (w_m)_t \sigma_j dx + B[w_m, \sigma_j; t] \\
& \quad = \int_{\Omega} f(x, t) \sigma_j dx. \quad (11)
\end{aligned}$$

Multiplying (11) by $(d_j^m)_t(t)$ and then summing over j , we obtain

$$\begin{aligned}
& \int_{\Omega} (w_m)_{tt}(t) (w_m)_t(t) dx + \int_{\Omega} u(x, t) ((w_m)_t)^2 dx \\
& \quad + B[w_m, (w_m)_t; t] = \int_{\Omega} f(x, t) (w_m)_t(t) dx. \quad (12)
\end{aligned}$$

Rewriting above equality and using Poincare's inequality Evans (1998), we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} ((w_m)_t)^2(t) dx + \int_{\Omega} (w_m)_x (w_m)_x(t) dx \right] \\
& \quad + \int_{\Omega} u(x, t) ((w_m)_t)^2 dx \\
& = \int_{\Omega} v(x, t) w_m(t) (w_m)_t(t) dx + \int_{\Omega} f(x, t) (w_m)_t(t) dx \\
& \leq \frac{N}{2} \left(\int_{\Omega} w_m^2(t) dx + \int_{\Omega} ((w_m)_t)^2(t) dx \right) \\
& \quad + \frac{1}{2} \int_{\Omega} ((w_m)_t)^2(t) dx + \frac{1}{2} \int_{\Omega} f^2(x, t) dx \\
& \leq C_0 \int_{\Omega} (w_m)_x (w_m)_x(t) dx + C_0 \int_{\Omega} ((w_m)_t)^2(t) dx \\
& \quad + \frac{1}{2} \int_{\Omega} f^2(x, t) dx \quad (13)
\end{aligned}$$

where C_0 is a constant independent of m . Gronwall's inequality Evans (1998) leads to the estimate

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left[\int_{\Omega} (((w_m)_t)^2(t) + |(w_m)_x|^2) dx \right] \\
& \leq C_1 \left[\int_{\Omega} ((w_m)_t)^2(0) dx + \int_{\Omega} |(w_m)_x(0)|^2 dx + \int_Q f^2 dx dt \right].
\end{aligned}$$

Thus we need to estimate the bounds for

$$\int_{\Omega} ((w_m)_t)^2(0) dx + \int_{\Omega} |(w_m)_x(0)|^2 dx.$$

We have

$$\begin{aligned}
\|(w_m)_t(0)\|_{L^2(\Omega)}^2 &= \left\| \sum_{j=1}^m (w_1, \sigma_j)_{L^2(\Omega)} \sigma_j(x) \right\|_{L^2(\Omega)}^2 \\
&\leq \left| \sum_{j=1}^m (w_1, \sigma_j)_{L^2(\Omega)} \right| \\
&\leq \|w_1\|_{L^2(\Omega)};
\end{aligned}$$

and

$$(w_m)_x(0) = \sum_{j=1}^m d_j^m(0) (\sigma_j)_x(x),$$

$$\begin{aligned}
\int_{\Omega} (w_m)_x(0)(w_m)_x(0)dx &= - \int_{\Omega} w_m(w_m)_{xx}(0)dx \\
&= - \int_{\Omega} \left(\sum_{j=1}^m d_j^m(0)\sigma_j(x) \right) \left(\sum_{j=1}^m d_j^m(0)(\sigma_j)_{xx}(x) \right) dx \\
&= - \int_{\Omega} \sum_{j=1}^m \lambda_j (d_j^m(0))^2 \sigma_j(x)^2 dx = \sum_{j=1}^m \lambda_j (d_j^m(0))^2 \\
&= \sum_{j=1}^m \lambda_j (w_0, \sigma_j)^2 \leq \|w_0\|_{H^1(\Omega)}.
\end{aligned}$$

For the last inequality above refer Evans (1998).

Thus, we have

$$\int_{\Omega} ((w_m)_t)^2(0)dx + \int_{\Omega} |(w_m)_x(0)|^2 dx \leq C_2(\|w_1\|_{L^2(\Omega)} + \|w_0\|_{H^1(\Omega)}).$$

So, we conclude that

$$\begin{aligned}
\sup_{0 \leq t \leq T} \left[\int_{\Omega} (((w_m)_t)^2(t) + |(w_m)_x|^2) dx \right] \\
\leq C_3 [\|w_1\|_{L^2(\Omega)} + \|w_0\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}]. \quad (14)
\end{aligned}$$

From (11), we obtain

$$\begin{aligned}
\int_{\Omega} (w_m)_{tt}(t)\phi dx + \int_{\Omega} u(x,t)(w_m)_t\phi dx + B[w_m, \phi; t] = \\
\int_{\Omega} f(x,t)\phi dx
\end{aligned}$$

for any $\phi \in H_0^1(\Omega)$. Thus,

$$\begin{aligned}
\left| \int_{\Omega} (w_m)_{tt}(t)\phi dx \right| &\leq \left| \int_{\Omega} u(x,t)(w_m)_t\phi dx \right| + |B[w_m, \phi; t]| \\
&\quad + \left| \int_{\Omega} f(x,t)\phi dx \right|.
\end{aligned}$$

Poincare's inequality and inequality (14) together gives

$$\left| \int_{\Omega} (w_m)_{tt}(t)\phi dx \right| \leq C_4 \|\phi\|_{H_0^1(\Omega)},$$

for any $\phi \in H_0^1(\Omega)$. Passing to the limit on a subsequence as $m \rightarrow \infty$, we obtain $w \in L^2([0, T]; H^1(\Omega))$ such that

$$\begin{aligned}
w_m &\longrightarrow w \text{ weakly in } L^2([0, T]; H^1(\Omega)), \\
(w_m)_t &\longrightarrow w_t \text{ weakly in } L^2(Q), \\
(w_m)_{tt} &\longrightarrow w_{tt} \text{ weakly in } L^2([0, T]; (H^1)^*(\Omega)).
\end{aligned}$$

Step 3. Consider the sequences $\{w_{0n}\} \subset H^2(\Omega)$, $\{w_{1n}\} \subset H_0^1(\Omega)$ and $\{u_n\} \subset U$ such that

$$\begin{aligned}
w_{0n} &\longrightarrow w_0 \text{ strongly in } H_0^1(\Omega), \\
w_{1n} &\longrightarrow w_1 \text{ strongly in } L^2(Q), \\
u_n &\longrightarrow u \text{ strongly in } L^2(Q),
\end{aligned}$$

Let w_n be the weak solution of (1) corresponding to the initial values w_{0n} , w_{1n} and control u_n , then w_n satisfies the regularity of previous step. Multiply the equation (1) by $(w_n)_t$ and integrate over $Q_t = \Omega \times (0, t)$ with $0 \leq t \leq T$, we obtain

$$\begin{aligned}
0 &= \int_{Q_t} [(w_n)_{tt}(w_n)_t + u((w_n)_t)^2 - (w_n)_{xx}(w_n)_t] dx d\tau \\
&\quad - \int_{Q_t} [vw_n(w_n)_t + f(w_n)_t] dx d\tau \\
&= \int_{Q_t} \left[\frac{1}{2} \frac{d}{dt} (((w_n)_t)^2 + |(w_n)_x|^2) + u((w_n)_t)^2 \right] dx d\tau \\
&\quad - \int_{Q_t} [vw_n(w_n)_t + f(w_n)_t] dx d\tau, \quad (15)
\end{aligned}$$

where for $0 \leq t \leq T$, $w_n(0, t) = w_n(l, t) = 0$, $(w_n)_t(0, t) = (w_n)_t(l, t) = 0$. From (15), we obtain

$$\begin{aligned}
\int_{\Omega \times \{t\}} [((w_n)_t)^2 + |(w_n)_x|^2] dx d\tau \\
= \int_{\Omega} [(w_{1n})^2 + |(w_{0n})_x|^2] dx \\
+ 2 \int_{Q_T} [-u((w_n)_t)^2 + vw_n(w_n)_t + f(w_n)_t] dx d\tau, \\
\leq \|w_{1n}\|_{L^2(\Omega)}^2 + \|(w_{0n})_x\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2 \\
+ \|(w_n)_t\|_{L^2(Q_t)}^2 + C_5(\|w_n\|_{L^2(Q_t)}^2 + \|(w_n)_t\|_{L^2(Q_t)}^2) \quad (16)
\end{aligned}$$

where constant C_5 depends only on M . Using Poincare's inequality, we obtain

$$\begin{aligned}
\int_{\Omega \times \{t\}} [((w_n)_t)^2 + (w_n)^2] dx \\
\leq C_6 \int_{Q_t} [((w_n)_t)^2 + (w_n)^2] dx d\tau \\
+ C_7(\|w_{1n}\|_{L^2(\Omega)}^2 + \|(w_{0n})_x\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2) \quad (17)
\end{aligned}$$

where constants C_6, C_7 are independent of w_n . Applying Gronwall's inequality to (17), we obtain

$$\begin{aligned}
\int_{\Omega \times \{t\}} [((w_n)_t)^2 + (w_n)^2] dx \\
\leq (1 + C_6 T e^{C_6 T}) C_7(\|w_{1n}\|_{L^2(\Omega)}^2 \\
+ \|(w_{0n})_x\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2) \quad (18)
\end{aligned}$$

Letting $n \rightarrow \infty$ and using inequalities (16), (18), we obtain

$$\begin{aligned}
\|w_t(t)\|_{L^2(\Omega)}^2 + \|(w)_x(t)\|_{L^2(\Omega)}^2 + \|w(t)\|_{L^2(\Omega)}^2 \\
\leq C_8(\|w_1\|_{L^2(\Omega)}^2 + \|(w_0)_x\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q_t)}^2) \quad (19)
\end{aligned}$$

where constant C_8 depends on Ω and T . Thus,

$$\begin{aligned}
\sup_{0 \leq t \leq T} (\|w(t)\|_{H_0^1(\Omega)}^2 + \|w_t(t)\|_{L^2(\Omega)}^2) \\
\leq C_8(\|w_1\|_{L^2(\Omega)}^2 + \|w_0\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2([0, T]; L^2(\Omega))}^2). \quad (20)
\end{aligned}$$

Next, we have

$$\langle w_{tt}, \phi \rangle(t) = - \int_{\Omega} [w_x \phi_x + u w_t \phi - v w \phi - f \phi](t) dx$$

for any $\phi \in H_0^1(\Omega)$. Let $\|\phi\|_{H_0^1(\Omega)} = 1$, then

$$|\langle w_{tt}, \phi \rangle(t)| \leq C_9(\|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2) \quad (21)$$

where constant C_9 depends on M . Using (21), we obtain

$$\begin{aligned}
\|w_{tt}\|_{L^2([0, T]; H^{-1}(\Omega))}^2 &\leq C_{10} \int_0^T (\|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H_0^1(\Omega)}^2) dt \\
&\quad + \int_0^T \|f\|_{L^2(\Omega)}^2 dt \\
&\leq C_{11} T (\|w_t\|_{L^2(\Omega)}^2 + \|w\|_{H_0^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2) \quad (22)
\end{aligned}$$

where C_{10} , C_{11} are constants which do not depend of u and v .

Inequalities (20) and (23) together imply the inequality (4). ■

Now, we prove the existence of optimal controls.

Theorem 2.2. (Existence of optimal controls). There exist an optimal controls $u^* \in U$ which minimizes the objective functional $J(u)$ for $u \in U$.

Proof. Let $\{u^n\} \in U$ be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(u^n) = \inf_{u \in U} J(u).$$

Let w^n be a solution of equation (1) with control u^n . Denote $w^n = w(u^n)$. By Theorem 2.1 we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|w^n(t)\|_{H_0^1(\Omega)} + \|(w^n)_t(t)\|_{L^2(\Omega)} \right) \\ & \quad + \|(w^n)_{tt}(t)\|_{L^2([0,T];H^{-1}(\Omega))} \\ & \leq C \left(\|f\|_{L^2([0,T];L^2(\Omega))} + \|w_0\|_{H_0^1(\Omega)} + \|w_1\|_{L^2(\Omega)} \right), \end{aligned}$$

where C is a constant.

On a subsequence, by weak compactness, there exists w^* in $C([0,T];H_0^1(\Omega))$ such that

$$\begin{aligned} w^n &\rightharpoonup w^* \quad \text{weakly in } L^\infty([0,T];H_0^1(\Omega)), \\ (w^n)_t &\rightharpoonup (w^*)_t \quad \text{weakly in } L^\infty([0,T];L^2(\Omega)), \\ (w^n)_{tt} &\rightharpoonup (w^*)_{tt} \quad \text{weakly in } L^\infty([0,T];H^{-1}(\Omega)), \\ u^n &\rightharpoonup u^* \quad \text{weakly in } H^1(Q), \\ (u^n)_x &\rightharpoonup (u^*)_x \quad \text{weakly in } L^2(Q), \\ (u^n)_t &\rightharpoonup (u^*)_t \quad \text{weakly in } L^2(Q). \end{aligned}$$

Using compactness result from Simon (1987), we have $w^n \rightarrow w^*$ strongly in $L^\infty([0,T];L^2(\Omega))$. By definition of weak solution, we have

$$\begin{aligned} & \int_0^T \langle (w^n)_{tt}, \phi \rangle dt \\ & = - \int_Q [(w^n)_x \phi_x + u^n ((w^n)_t) \phi - w^n \phi - f \phi] dx dt. \end{aligned}$$

We have,

$$\begin{aligned} & \left| \int_0^T \int_0^l (u^n (w^n)_t \phi - u^* (w^*)_t \phi) dx dt \right| \\ & \leq \int_0^T \int_0^l |(u^n - u^*) \phi| |(w^n)_t| dx dt \\ & \quad + \left| \int_0^T \int_0^l u^* \phi ((w^n)_t - (w^*)_t) dx dt \right| \\ & \leq \left(\int_0^T \int_0^l |(u^n - u^*) \phi|^2 dx dt \right)^{1/2} \\ & \quad \left(\int_0^T \int_0^l |(w^n)_t|^2 dx dt \right)^{1/2} \\ & \quad + \left| \int_0^T \int_0^l u^* \phi ((w^n)_t - (w^*)_t) dx dt \right| \\ & \leq C_1 \left(\int_0^T \int_0^l |u^n - u^*|^2 dx dt \right)^{1/2} \\ & \quad \left(\int_0^T \int_0^l |(w^n)_t|^2 dx dt \right)^{1/2} \\ & \quad + \left| \int_0^T \int_0^l u^* \phi ((w^n)_t - (w^*)_t) dx dt \right| \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where C_1 depends on the L^∞ bound on ϕ .

The first term converges, since $u^n \rightharpoonup u^*$ weakly in $H^1(Q)$ and $\{(w^n)_t\}$ is uniformly bounded in $L^2(Q)$. Also, $u^* \phi \in L^2(Q)$ and $(w^n)_t \rightharpoonup (w^*)_t$ weakly in $L^\infty([0,T];L^2(\Omega))$, the second term converges. Thus,

$$u^n (w^n)_t \rightharpoonup u^* (w^*)_t \text{ weakly in } L^2(Q).$$

Passing to the limit as $n \rightarrow \infty$ in the weak formulation of w^n , we obtain

$$\begin{aligned} & \int_0^T \langle (w^*)_{tt}, \phi \rangle dt \\ & = - \int_Q [(w^*)_x \phi_x + u^* ((w^*)_t) \phi - w^* \phi - f \phi] dx dt. \end{aligned}$$

Thus $w^* = w(u^*)$ is the solution of state equation (1) with controls u^* and v^* . Since

$$\begin{aligned} J(u^*) &= \frac{\alpha}{2} \int_Q (w^* - z)^2 dx dt \\ & \quad + \frac{\beta}{2} \int_Q [((u^*)_x)^2 + ((u^*)_t)^2] dx dt. \end{aligned}$$

Using lower-semicontinuity of L^2 norm with respect to weak convergence, we have

$$\begin{aligned} J(u^*) &\leq \lim_{n \rightarrow \infty} \frac{\beta}{2} \int_Q [((u^n)_x)^2 + ((u^n)_t)^2] dx dt \\ & \quad + \lim_{n \rightarrow \infty} \frac{\alpha}{2} \int_Q (w^n - z)^2 dx dt \\ &\leq \lim_{n \rightarrow \infty} \inf J(u^n) \\ &= \inf_{u \in U} J(u). \end{aligned}$$

Thus, we conclude that u^* is an optimal control. ■

3. CHARACTERIZATION OF THE OPTIMAL CONTROL

By differentiating the objective functional $J(u)$ with respect to the control u , we derive the optimality system. As $w = w(u)$ is involved in $J(u)$, we must first prove the appropriate differentiability of the mapping

$$u \rightarrow w(u).$$

Lemma 3.1. The mapping

$$u \in U \rightarrow \tilde{w}(u) = (w(u), w_t(u)) \in L^2([0, T]; \mathcal{H})$$

is differentiable in the following sense:

$$\frac{\tilde{w}(u + \epsilon h) - \tilde{w}(u)}{\epsilon} \rightharpoonup \tilde{\Psi}$$

weakly in $L^2([0, T]; \mathcal{H})$ as $\epsilon \rightarrow 0$, for any u satisfying $u + \epsilon h \in U$ for small ϵ and $h \in H_0^1(Q)$.

Moreover $\tilde{\Psi} = (\Psi, \Psi_t) \in L^2([0, T]; \mathcal{H})$ is a weak solution of the following problem:

$$\begin{aligned} \Psi_t &= \Psi_{xx} + u\Psi_t + v\Psi - hw_t, \text{ in } Q, \\ \Psi &= 0, & x = 0, \quad t \in (0, T), \\ \Psi &= 0, & x = l, \quad t \in (0, T), \\ \Psi &= 0, & t = 0, \quad x \in \Omega, \\ \Psi_t &= 0, & t = 0, \quad x \in \Omega, \end{aligned} \quad (23)$$

where $w = w(u)$.

Proof. Let $w^\epsilon = w(u + \epsilon l)$ and $w = w(u)$, then $(w^\epsilon - w)/\epsilon$ is a weak solution of

$$\begin{aligned} \left(\frac{w^\epsilon - w}{\epsilon} \right)_{tt} &= \left(\frac{w^\epsilon - w}{\epsilon} \right)_{xx} + u \left(\frac{w^\epsilon - w}{\epsilon} \right)_t \\ &\quad + v \left(\frac{w^\epsilon - w}{\epsilon} \right) - hw_t^\epsilon, \text{ in } Q, \\ \frac{w^\epsilon - w}{\epsilon} &= 0, \quad x = 0, \quad t \in (0, T), \\ \frac{w^\epsilon - w}{\epsilon} &= 0, \quad x = l, \quad t \in (0, T), \\ \frac{w^\epsilon - w}{\epsilon} &= 0, \quad t = 0, \quad x \in \Omega, \\ \left(\frac{w^\epsilon - w}{\epsilon} \right)_t &= 0, \quad t = 0, \quad x \in \Omega. \end{aligned}$$

Using Theorem 2.1, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \left(\left\| \frac{w^\epsilon - w}{\epsilon} \right\|_{H_0^1(\Omega)} + \left\| \frac{w_t^\epsilon - w_t}{\epsilon} \right\|_{L^2(\Omega)} \right) \\ + \left\| \frac{w_{tt}^\epsilon - w_{tt}}{\epsilon} \right\|_{L^2([0, T]; H^{-1}(\Omega))} \\ \leq C \|hw_t^\epsilon\|_{L^2([0, T]; L^2(\Omega))} \\ \leq C_1 \|h\|_{L^\infty(Q)} \sup_{0 \leq t \leq T} \|w_t^\epsilon\|_{L^2(\Omega)} \leq C_3 \end{aligned}$$

where C_3 is a constant. Hence on a subspace, by weak compactness, we have

$$\begin{aligned} \frac{w_{tt}^\epsilon - w_{tt}}{\epsilon} &\rightharpoonup \Psi_{tt} \text{ weakly in } L^\infty([0, T]; H^{-1}(\Omega)), \\ \frac{w_t^\epsilon - w_t}{\epsilon} &\rightharpoonup \Psi_t \text{ weakly in } L^\infty([0, T]; L^2(\Omega)), \\ \frac{w^\epsilon - w}{\epsilon} &\rightharpoonup \Psi \text{ weakly in } L^\infty([0, T]; H_0^1(\Omega)). \end{aligned}$$

By the definition of weak solution, we have

$$\begin{aligned} \left\langle \frac{w_{tt}^\epsilon - w_{tt}}{\epsilon}, \phi \right\rangle + \int_\Omega \left(\frac{w^\epsilon - w}{\epsilon} \right)_x \phi_x dx \\ = - \int_\Omega u \left(\frac{w^\epsilon - w}{\epsilon} \right)_t \phi dx + \int_\Omega v \left(\frac{w^\epsilon - w}{\epsilon} \right) \phi dx \\ - \int_\Omega hw_t^\epsilon \phi dx \end{aligned}$$

for any $\phi \in H_0^1(\Omega)$, a.e. $0 \leq t \leq T$.

Letting $\epsilon \rightarrow 0$ and using $w^\epsilon \rightarrow w$, $w_t^\epsilon \rightarrow w_t$ weakly in $L^2(\Omega)$, we obtain that Ψ is the weak solution of (23). ■

Now we prove the necessary conditions that characterize an optimal control.

Theorem 3.1. Given an optimal control u and corresponding $\tilde{w} = \tilde{w}(u) = (w, w_t)$, there exists a weak solution $\tilde{p} = (p, p_t)$ in $\mathcal{H}$ to the adjoint problem:

$$\begin{aligned} p_{tt} &= p_{xx} + (up)_t + vp + (w - z) \text{ in } Q, \\ p &= 0, & x = 0, \quad t \in (0, T), \\ p &= 0, & x = l, \quad t \in (0, T), \\ p &= 0, & t = T, \quad x \in \Omega, \\ p_t &= 0, & t = T, \quad x \in \Omega. \end{aligned} \quad (24)$$

Furthermore, u satisfies the following variational inequality:

$$\min(\max(-(u_{tt} + u_{xx} + \frac{\alpha}{\beta}pw_t), u - M), u + M) = 0, \quad (25)$$

where $w = w(u)$.

Proof. Let $u \in U$ be an optimal control and $w = w(u)$ be the corresponding optimal solution. Let $u + \epsilon h \in U$ for $\epsilon > 0$ and $w^\epsilon = w(u + \epsilon h)$ be the corresponding weak solution of equation (1). We compute the directional derivative of the objective functional $J(u)$ with respect to u in the direction of h . Since J is supposed to attain its minimum for u , we have

$$\begin{aligned} 0 &\leq \lim_{\epsilon \rightarrow 0^+} \frac{J(u + \epsilon h) - J(u)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0^+} \left\{ \frac{\alpha}{2\epsilon} \int_Q [(w^\epsilon - z)^2 - (w - z)^2] dx dt \right\} \\ &\quad + \lim_{\epsilon \rightarrow 0^+} \left\{ \frac{\beta}{2\epsilon} \int_Q [(u_t + \epsilon h_t)^2 - (u_t)^2] dx dt \right\} \\ &\quad + \lim_{\epsilon \rightarrow 0^+} \left\{ \frac{\beta}{2\epsilon} \int_Q [(u_x + \epsilon h_x)^2 - (u_x)^2] dx dt \right\} \\ &= \alpha \int_Q \Psi(w - z) dx dt + \beta \int_Q [u_x h_x + u_t h_t] dx dt, \end{aligned}$$

where Ψ is as defined in Lemma 3.1. We need to prove the existence of the solution to the adjoint equation (24). It does not follow from the existence and uniqueness of solution of the state equation (1) as adjoint equation contains the term $(up)_t$.

We write the adjoint equation in semigroup form to as

$$\frac{d}{dt} \tilde{p}(t) = A\tilde{p}(t) + \begin{pmatrix} 0 \\ (up)_t + vp + w - z \end{pmatrix},$$

where A generates a unitary group on $\mathcal{H}$.

We expect a solution of (24) of the form

$$\tilde{p}(t) = \int_t^T e^{A(t-s)} \begin{pmatrix} 0 \\ (up)_t + vp + w - z \end{pmatrix} ds.$$

Integration by parts gives,

$$\begin{aligned}\tilde{p}(t) = & \int_t^T \left(-Ae^{A(t-s)} \begin{pmatrix} 0 \\ up \end{pmatrix} \right) ds \\ & + \int_t^T \left(e^{A(t-s)} \begin{pmatrix} 0 \\ vp + w - z \end{pmatrix} \right) ds + \begin{pmatrix} 0 \\ up(t) \end{pmatrix}.\end{aligned}$$

Now consider the map

$$\begin{aligned}F\tilde{p}(t) = & \int_t^T \left(-Ae^{A(t-s)} \begin{pmatrix} 0 \\ up \end{pmatrix} \right) ds \\ & + \int_t^T \left(e^{A(t-s)} \begin{pmatrix} 0 \\ vp + w - z \end{pmatrix} \right) ds + \begin{pmatrix} 0 \\ up(t) \end{pmatrix}.\end{aligned}$$

Using technique similar to the proof of Theorem 2.1, step 1, we can prove that the map F has a unique fixed point on $[0, T]$, which guarantees the existence and uniqueness of weak solution to the adjoint equation.

Assume that p is the weak solution of the problem (24), i.e., p satisfies

$$\begin{aligned}\langle p_{tt}, \phi \rangle(t) + \int_{\Omega} p_x \phi_x(t) dx = & \int_{\Omega} (up)_t \phi dx + \int_{\Omega} vp \phi dx \\ & + \int_{\Omega} (w - z) \phi(t) dx\end{aligned}$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \leq t \leq T$. Then

$$\begin{aligned}0 \leq & \alpha \int_Q \Psi(w - z) dx dt + \beta \int_Q [u_x h_x + u_t h_t] dx dt \\ = & \alpha \int_0^T \left\{ \langle p_{tt}, \Psi \rangle(t) + \int_{\Omega} p_x \Psi_x(t) dx - \int_{\Omega} (up)_t \Psi \right\} dt dx \\ & - \alpha \int_0^T \left\{ \int_{\Omega} vp \Psi dx \right\} dt + \beta \int_Q [u_x h_x + u_t h_t] dx dt \\ = & \alpha \int_0^T \left\{ \int_{\Omega} -hw_t p dx \right\} dt + \beta \int_Q [u_x h_x + u_t h_t] dx dt \\ = & \int_Q \{ -\alpha hw_t p + \beta [u_x h_x + u_t h_t] \} dx dt.\end{aligned}$$

The last expression gives the variational inequality (25). ■

We obtain the optimality system consisting of two hyperbolic PDEs and an variational inequality, (1), (24), and (25).

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